Physics (H)- SEM-II_CC IV: WAVES AND OPTICS Online Class-3

Topics to Cover: Linearity and Superposition Principle. Superposition of two collinear oscillations having (1) equal frequencies and (2) different frequencies (Beats). Superposition of N collinear Harmonic Oscillations with (1) equal phase differences and (2) equal frequency differences.

Already discussed in class: a) Linearity and Superposition Principle. b) Superposition of two collinear oscillations having equal frequencies and different frequencies (Beats).

Today we'll discuss the resultant motion due to superposition of a large number of simple harmonic motions of same amplitude (a) and same frequency (ω) along the x-axis but increasing progressively in phase by the phase angle φ.

Let the superposing simple harmonic motions are given by

$X_1 = a \sin \omega t$,

$X_2 = a \sin (\omega t + \delta)$,

$X_3 = a \sin (\omega t + 2\delta)$,

…..

…..

$X_N = a \sin [\omega t + (N-1)\delta]$.

Hence, resultant oscillation is given by, $X = \Sigma X_i$

Vector addition method:

The resultant amplitude may be obtained by the vector polygon method (see the figure). The polygon OABCO is drawn with each side of length a and making an angle δ with the neighbouring side. Respective oscillations are shown by chords of equal length = a of a circle of radius r (say). The resultant has the amplitude AC with the phase angle $\angle CAB = \alpha$ with respect to the first vibration.

The first oscillation, $X_1 = a \sin \omega t$, is depicted by chord AB and resultant of superposition of all the N oscillations is depicted by chord AC of length R and resultant oscillation makes an angle $\angle CAB = \alpha$ with the first oscillation AB.

We have to find out R and α

As each chord makes an equal angle δ at the centre O, hence angle subtended by chord AB, $\angle AOB = \delta$, and that by the chord AC, $\angle AOC = N\delta$

Now from the geometry of the figure we have AB $= a = 2r\sin\frac{\delta}{2}$ and the resultant displacement

AC = $R = 2r\sin\frac{N\delta}{2}$

Hence, $R = a\frac{\sin\frac{N\delta}{2}}{\sin\frac{\delta}{2}}$

Also, $\angle OAB = \frac{\pi}{2} - \frac{\delta}{2}$

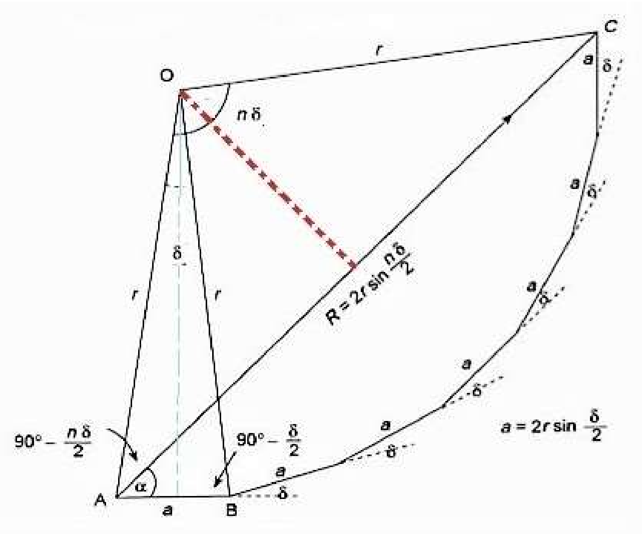
$\angle OAC = \frac{\pi}{2} - \frac{N\delta}{2}$

And $\alpha = \angle CAB = \angle OAB - \angle OAC = \frac{(N-1)\delta}{2}$

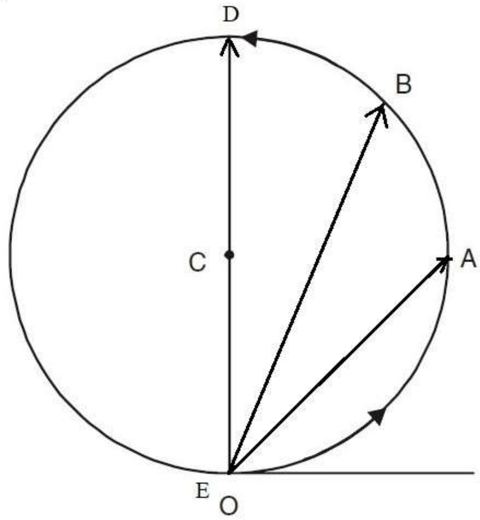
Hence, resultant oscillation is given by,

$X = \Sigma X_i$

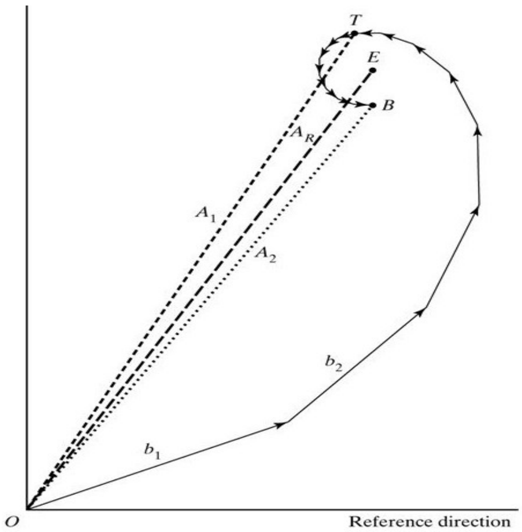
$= R\sin(\omega t + \alpha) = a\frac{\sin\frac{N\delta}{2}}{\sin\frac{\delta}{2}}\sin\left[\omega t + \frac{(N-1)\delta}{2}\right]$;



Note 1: If the number of superposing vibrations is very large (i.e., $N \to \alpha$) and amplitude becomes very small (*i.e., $a \to 0$*) but phase δ increases continuously in same interval then the polygon will become an arc of a circle and the chord joining the first and the last points of the arc will represent the amplitude of the resultant vibration (see figure). When the last component vibration is at *A* or at B then the resultant vibration will be given by vector $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. But if tip of the last vibration be ad D then the first and the last component vibration are in opposite phase and the amplitude of the resultant vibration, $\overrightarrow{OD}$ = diameter of the circle. When the last component vibration E is at *O*, the first and the last component vibrations are in phase, the polygon becomes a complete circle and the amplitude of the resultant vibration is zero.



Note 2: When the successive amplitudes of a large number of component vibrations decrease slowly and the phase angles increase continuously the polygon becomes a spiral called Cornu spiral



Analytical method:

Let the superposing simple harmonic motions are given by

$X_1 = a \sin \omega t$,

$X_2 = a \sin (\omega t + \delta)$,

$X_3 = a \sin (\omega t + 2\delta)$,

…..

…..

$X_N = a \sin [\omega t + (N-1)\delta]$.

Hence, resultant oscillation is given by, $X = \Sigma X_i$

$= a \sin \omega t\, [1 + \cos\delta + \cos 2\delta + ... + \cos (N-1)\delta] + a \cos \omega t\, [0 + \sin\delta + \sin 2\delta + ... + \sin (N-1)\delta]$,

$= R \cos\theta \sin\omega t + R \sin\theta \cos\omega t$

$= R \sin (\omega t + \theta)$

Where, $R \cos\theta = a\,[1 + \cos\delta + \cos 2\delta + ... + \cos (N-1)\delta]$,

And, $R \sin\theta = a\,[0 + \sin\delta + \sin 2\delta + ... + \sin (N-1)\delta]$

We have to find out values of $R \cos\theta = a\,[1 + \cos\delta + \cos 2\delta + ... + \cos (N-1)\delta]$,

and $R \sin\theta = a\,[0 + \sin\delta + \sin 2\delta + ... + \sin (N-1)\delta]$

To do this we have to use complex algebra, where we can express $e^{i\emptyset} = cos\emptyset + isin\emptyset$

also we can write,

$$\cos\delta + \cos 2\delta + ... + \cos (N-1)\delta = \text{Real part of } (e^{i\delta} + e^{i2\delta} + e^{i3\delta} + \cdots + e^{i(N-1)\delta})$$

And,

$$\sin\delta + \sin 2\delta + ... + \sin (N-1)\delta = \text{Imaginary part of } (e^{i\delta} + e^{i2\delta} + e^{i3\delta} + \cdots + e^{i(N-1)\delta})$$

Now,

$$e^{i\emptyset} + e^{i2\emptyset} + e^{i3\emptyset} + \cdots + e^{i(N-1)\emptyset}\text{ ; here } \varphi \text{ is any arbitrary angle.}$$

$$= e^{i\emptyset} + (e^{i\emptyset})^2 + (e^{i\emptyset})^3 + \cdots + (e^{i\emptyset})^{(N-1)}$$

[This is a G.P. series of (N-1) terms having 1st term = $e^{i\emptyset}$ and common ratio = $e^{i\emptyset}$, and $e^{i\emptyset} \le 1$]

$$= \frac{e^{i\emptyset}\{1 - \left(e^{i\emptyset}\right)^{(N-1)}\}}{1 - e^{i\emptyset}}$$

$$= \frac{e^{i\emptyset}\left(e^{i\emptyset}\right)^{(\frac{N-1}{2})}\{\left(e^{i\emptyset}\right)^{-(\frac{N-1}{2})} - \left(e^{i\emptyset}\right)^{(\frac{N-1}{2})}\}}{e^{i\frac{\emptyset}{2}}\left(e^{-i\frac{\emptyset}{2}} - e^{i\frac{\emptyset}{2}}\right)}$$

$$= e^{(i\emptyset\frac{N}{2})}\frac{\{\frac{e^{i\emptyset(\frac{N-1}{2})}-e^{-i\emptyset(\frac{N-1}{2})}}{2i}\}}{\frac{(e^{i\frac{\emptyset}{2}}-e^{-i\frac{\emptyset}{2}})}{2i}}$$

$$= (cos\frac{N\emptyset}{2}+isin\frac{N\emptyset}{2})\frac{sin(\frac{N-1}{2})\emptyset}{sin\frac{\emptyset}{2}}$$

$$= \left(cos\frac{N\emptyset}{2}\right)\frac{\sin\left(\frac{N-1}{2}\right)\emptyset}{sin\frac{\emptyset}{2}} + (isin\frac{N\emptyset}{2})\frac{sin(\frac{N-1}{2})\emptyset}{sin\frac{\emptyset}{2}}$$

Separating real and imaginary part of the above expression we get,

$$\cos\varphi + \cos 2\varphi + \ldots. + \cos(N-1)\varphi = \left(cos\frac{N\emptyset}{2}\right)\frac{sin\left(\frac{N-1}{2}\right)\emptyset}{sin\frac{\emptyset}{2}}$$

$$\text{and,}\ \sin\varphi + \sin 2\varphi + \ldots. + \sin(N-1)\varphi = (sin\frac{N\emptyset}{2})\frac{sin(\frac{N-1}{2})\emptyset}{sin\frac{\emptyset}{2}}$$

$$\text{Now, } R\cos\theta = a[1+\cos\delta+\cos 2\delta+\cdots.+\cos(N-1)\delta]$$

$$= a[1+\left(cos\frac{N\delta}{2}\right)\frac{sin\left(\frac{N-1}{2}\right)\delta}{sin\frac{\delta}{2}}]$$

$$= a[\frac{sin\frac{\delta}{2}+\left(cos\frac{N\delta}{2}\right)sin\left(\frac{N-1}{2}\right)\delta}{sin\frac{\delta}{2}}]$$

$$= a[\frac{\sin\{N-(N-1)\}\frac{\delta}{2}+\left(cos\frac{N\delta}{2}\right)sin\left(\frac{N-1}{2}\right)\delta}{sin\frac{\delta}{2}}]$$

$$= a[\frac{sin\frac{N\delta}{2}cos\left(\frac{N-1}{2}\right)\delta-\left(cos\frac{N\delta}{2}\right)sin\left(\frac{N-1}{2}\right)\delta+\left(cos\frac{N\delta}{2}\right)sin\left(\frac{N-1}{2}\right)\delta}{sin\frac{\delta}{2}}]$$

$$= a[\frac{sin\frac{N\delta}{2}cos\left(\frac{N-1}{2}\right)\delta}{sin\frac{\delta}{2}}]$$

$R\sin\theta = a\,[0+\sin\delta+\sin 2\delta+\ldots.+\sin(N-1)\,\delta] = a[\frac{sin\frac{N\delta}{2}sin\left(\frac{N-1}{2}\right)\delta}{sin\frac{\delta}{2}}]$

Therefore,

$$R = \sqrt{(Rcos\theta)^2+(Rsin\theta)^2}$$

$$= a[\frac{sin\frac{N\delta}{2}}{sin\frac{\delta}{2}}]$$

And, $tan\theta = \frac{Rsin\theta}{Rcos\theta} = \frac{=a[\frac{sin\frac{N\delta}{2}sin\left(\frac{N-1}{2}\right)\delta}{sin\frac{\emptyset}{2}}]}{=a[\frac{sin\frac{N\delta}{2}cos\left(\frac{N-1}{2}\right)\delta}{sin\frac{\delta}{2}}]} = tan\left(\frac{N-1}{2}\right)\emptyset$

Or, $\theta = (\frac{N-1}{2})\delta$

Thus, we see, the resultant motion due to superposition of a large number of simple harmonic motions of same amplitude (*a*) and same frequency (*ω*) along the x-axis but increasing progressively in phase by the phase angle δ is a simple harmonic with amplitude and phase angle given by,

$$R = a[\frac{sin\frac{N\delta}{2}}{sin\frac{\delta}{2}}]$$

And,

$$\theta = (\frac{N-1}{2})\delta$$

When *N* is large and ∅ is small, we may write

$$\theta \approx \left(\frac{N}{2}\right)\delta \; or \; \frac{\theta}{N} \approx (\frac{\delta}{2})$$

In that situation resultant amplitude becomes $R = a\left[\frac{sin\theta}{sin\frac{\theta}{N}}\right] = a\left[\frac{sin\theta}{\frac{\theta}{N}}\right] = aN\left[\frac{sin\theta}{\theta}\right]$

and the phase difference between the first component vibration X_1 and N^{th} component vibration X_N is nearly equal to $N\delta = 2\theta$.